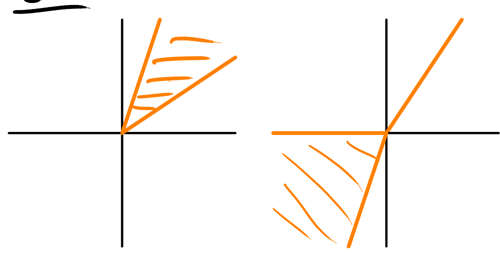
Primer to toric varieties

I Cones

$M \simeq \mathbb{Z}^n$ free abelian group of rk. n; $V = M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$

$N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual; $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) = N \otimes_{\mathbb{Z}} \mathbb{R}$

Ex:



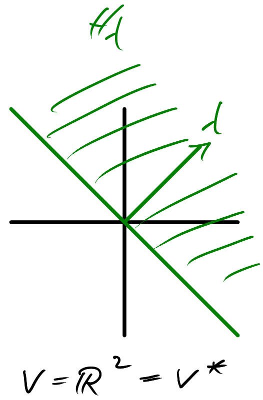
Def: • A <u>cone (in V)</u> is a subset $\sigma \subset V$ that is invariant under $\mathbb{R}_{\geq 0}$.

• A cone σ is

- <u>convex</u> if it is closed under addition ($\Leftrightarrow$ a convex subset of V);
- <u>strictly convex</u> if it does not contain a linear subspace other than $\{0\}$;
- <u>polyhedral</u> if there are $\lambda_1 - \lambda_n \in V^*$ s.t.

$$\sigma = \bigcap_{i=1}^{n} H_{\lambda_i} \quad \text{where} \quad \underbrace{H_{\lambda_i} = \{v \in V \mid \langle \lambda_i, v \rangle \geq 0\}}_{\text{half space}}$$

$(\langle \lambda, v \rangle := \lambda(v))$

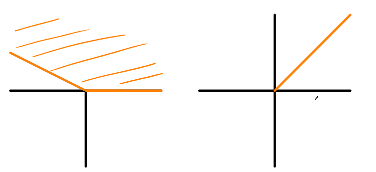


- <u>rational</u> if it is polyhedral and

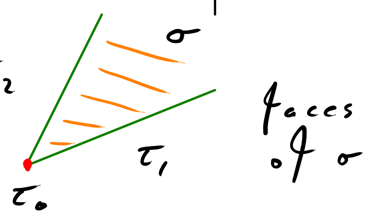
$$\sigma = \bigcap_{i=1}^{n} H_{\lambda_i}$$

for $\lambda_1 - \lambda_n \in N$.

$\rightarrow$ From now on, we assume that all cones are strictly convex & rational.



• A <u>face of σ</u> is an intersection of the form $\tau = \sigma \cap H_\lambda$ for some $\lambda \in N$.



II Affine toric varieties

- The <u>dual cone of σ</u> is

$$\sigma^\vee = \{ \lambda \in V^* \mid \langle \lambda, v \rangle \geq 0 \quad \forall v \in \sigma \}$$

<u>Rem:</u> In general, $\sigma^\vee$ is not strictly convex, but

$$\dim(\text{largest lin. subspace } L \subseteq \sigma^\vee) = \operatorname{codim}_V(\operatorname{span}(\sigma))$$

In particular $\{0\}^\vee = V^*$.

<u>Def:</u>
- Define the submonoid

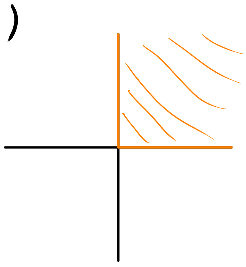
$$A_\sigma = \sigma^\vee \cap N \qquad \text{of the group } N,$$

which we write multiplicatively ("$T^{\lambda+\mu} = T^\lambda \cdot T^\mu$"), with unit 1 ($= T^0$).

- Let k be a ring (e.g. $k = \mathbb{C}$, $k = \mathbb{Z}$)

The affine toric k-variety associated with σ is $U_\sigma = \operatorname{Spec} k[A_\sigma]$.

<u>Ex:</u> (1)



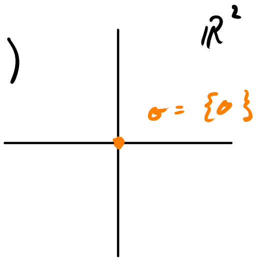
$\sigma = \mathbb{R}^2_{\geq 0} \leq \mathbb{R}^2 \rightsquigarrow \sigma^\vee = \{\lambda \in \mathbb{R}^2 \mid \langle \lambda, v \rangle \geq 0 \ \forall v \in \sigma\}$
$= \mathbb{R}^2_{\geq 0}$

$\left(\begin{matrix} M = \mathbb{Z}^2 \\ N = \operatorname{Hom}(\mathbb{Z}^2, \mathbb{Z}) = \mathbb{Z}^2 \end{matrix} \right)$

$\rightsquigarrow A_\sigma = \sigma^\vee \cap \mathbb{Z}^2 = \mathbb{N}^2$ (written multiplicatively)
$= \{ T_1^i T_2^j \mid i, j \in \mathbb{N} \}$

$\rightsquigarrow k[A_\sigma] = k[T_1, T_2]$
& $U_\sigma = \mathbb{A}^2_k$ (affine space)

(2)



$\rightsquigarrow \sigma^\vee = \{\lambda \in \mathbb{R}^2 \mid \langle \lambda, v \rangle \geq 0 \ \forall v \in \sigma\} = \mathbb{R}^2$

$\rightsquigarrow A_\sigma = \sigma^\vee \cap \mathbb{Z}^2 = \mathbb{Z}^2 = \{ T_1^i T_2^j \mid i, j \in \mathbb{Z} \}$

$\rightsquigarrow k[A_\sigma] = k[T_1^{\pm 1}, T_2^{\pm 1}]$

& $U_\sigma = \mathbb{G}^2_{m,k}$

<u>Fact:</u> $\dim U_\sigma = \dim_{\mathbb{R}} V$.

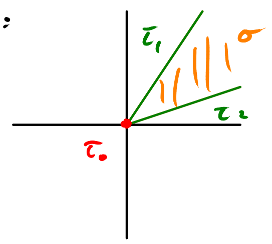
III Fans

Def: A fan (in V) is a collection Δ of cones in V

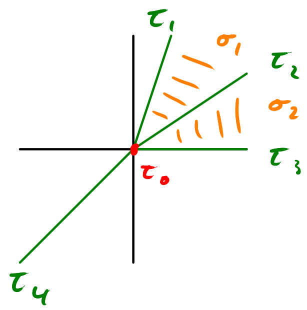
s.t. • every face τ of a cone $\sigma \in \Delta$ is in Δ;

• for all $\sigma, \tau \in \Delta$, $\sigma \cap \tau$ is a face of both σ and τ.

Ex:



$\Delta = \{\sigma, \tau_0, \tau_1, \tau_2\}$



$\Delta = \{\sigma_1, \sigma_2, \tau_0 - \tau_4\}$

Note: Δ always contains the trivial cone $\{0\}$.

IV Toric Varieties

Lemma: Δ fan
$\sigma \in \Delta$
τ face of σ

Then (1) $A_\sigma \subset A_\tau$ as subsets of N;

(2) $k[A_\tau] = S^{-1} k[A_\sigma]$ for $S = A_\tau^\times \cap A_\sigma$;

(3) $U_\tau \hookrightarrow U_\sigma$ is a principal open immersion.

Def: The toric variety associated with Δ is

$$X(\Delta) = \operatorname{colim}_{\sigma \in \Delta} U_\sigma = \text{"}\bigcup_{\sigma \in \Delta} U_\sigma\text{"}$$

Lemma: $\sigma, \tau \in \Delta$

Then • $U_\sigma \subset X(\Delta)$ is open dense;

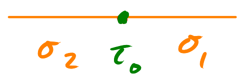
• $U_\tau \subset U_\sigma$ iff. $\tau \subset \sigma$

relative dim. over Spec k; • $\dim_k X(\Delta) = \dim_{\mathbb{R}}(V) =: n$

if k is a field, then this is simply $\dim X(\Delta)$

• $T = U_{\{0\}} \simeq \mathbb{G}_{m,k}^n \subset X(\Delta)$

$\Rightarrow X(\Delta)$ is a rational k-variety.

Ex: 

$$\rightsquigarrow \begin{cases} A_{\sigma_1} = \{T^i \mid i \in \mathbb{Z}_{\geq 0}\} \\ \cap \\ A_{\tau_0} = \{T^i \mid i \in \mathbb{Z}\} \\ \cup \\ A_{\sigma_2} = \{T^i \mid i \in \mathbb{Z}_{\leq 0}\} \end{cases} \rightsquigarrow \begin{cases} U_{\sigma_1} = \mathbb{A}^1_k \\ \uparrow \\ U_{\tau_0} = \mathbb{G}_{m,k} \\ \downarrow \\ U_{\sigma_2} = \mathbb{A}^1_k \end{cases}$$

$$\rightsquigarrow \quad X(\Delta) = \mathbb{A}^1_k \sqcup_{\mathbb{G}_{m,k}} \mathbb{A}^1_k = \mathbb{P}^1_k$$

V The torus action

$\rightarrow$ $T \circlearrowright U_\sigma$ by

$$k[A_\sigma] \longrightarrow k[A_{\{0\}}] \otimes_k k[A_\sigma],$$
$$T^a \longmapsto T^a \otimes T^a$$

which extends the group law $T \times T \to T$ to $T \times U_\sigma \to U_\sigma$; we get an action

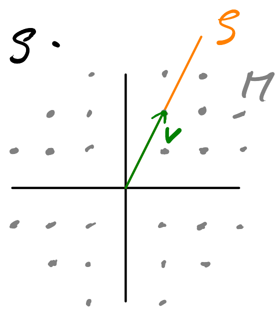
$$T \times X(\Delta) \to X(\Delta).$$

- For $\sigma \in \Delta$, define $\mathcal{O}(\sigma) = \operatorname{Spec} k[A_\sigma^\times]$;

Prop:
- $\dim_k \mathcal{O}(\sigma) = \dim(\operatorname{span}(\sigma))$
- The map $k[A_\sigma] \to k[A_\sigma^\times]$
$$a \mapsto \begin{cases} a & \text{if } a \in A_\sigma^\times \\ 0 & \text{if not} \end{cases}$$
defines a closed immersion $\mathcal{O}(\sigma) \hookrightarrow U_\sigma$;
- $T \circlearrowright U_\sigma$ restricts to an action of T on $\mathcal{O}(\sigma)$. $T(K) \circlearrowright \mathcal{O}(\sigma)(K)$ is transitive for every field K.
- $\Phi: \coprod_{\sigma \in \Delta} \mathcal{O}(\sigma) \longrightarrow X(\Delta)$ is a decomposition in the sense that it is a bijection on K-rational points for every field K.

VI Smooth & proper toric varieties

Def: A ray (in V) is a 1-dimensional cone ρ.



The primitive vector of ρ is the element $v \in (M \cap \rho) - \{0\}$ of smallest absolute value.

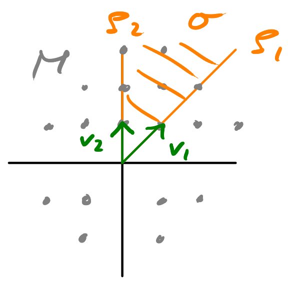
- σ cone

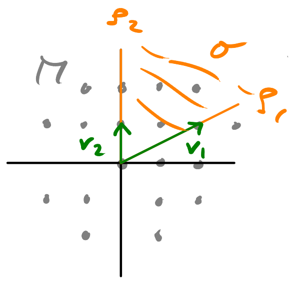
A ray of σ is a ray ρ that is a face of σ.

Thm: σ cone in V, $\rho_1 - \rho_s$ rays of σ with prim. vectors $v_1 - v_s$

Then U_σ is smooth over k iff. $(v_1 - v_s)$ can be extended to a basis of M.

Ex:





$U_\sigma \simeq \mathbb{A}^2_k$ smooth

$U_\sigma \simeq \operatorname{Spec} k[X,Y,Z]/\langle XY - Z^2 \rangle$ conic with singularity in (X,Y,Z).

Thm: Δ fan, k a field

Then $X(\Delta)$ is a normal k-variety.

In particular, $\operatorname{codim}(X(\Delta)^{\mathrm{sing}}) \geq 2$.

$\rightarrow$ Δ fan, $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma \subset V$ (support of Δ)

Thm: Δ fan

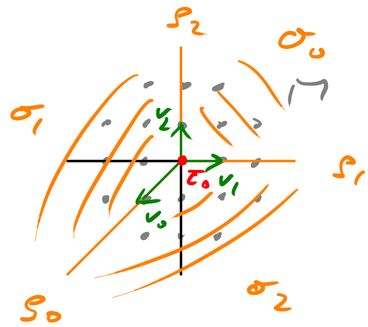
Then $X(\Delta)$ is proper over k iff. $|\Delta| = V$.

Ex: $v_1 = (1, 0, \dots, 0), \dots, v_n = (0, \dots, 0, 1), v_0 = (-1, \dots, -1) \in \mathbb{R}^n$

$\sigma_I := \{ \sum_{i \in I} c_i \cdot v_i \mid c_i \in \mathbb{R}_{\geq 0} \}$ for $I \subset \{0, \dots, n\}$

$\Delta = \{ \sigma_I \mid I \subsetneq \{0, \dots, n\} \}$

Then $X(\Delta) \simeq \mathbb{P}^n_k$.



fan of $\mathbb{P}^2_k$